

HOMOGENEITY DEGREE OF SOME SYMMETRIC PRODUCTS

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ABSTRACT. For a metric continuum X , we consider the n^{th} -symmetric product $F_n(X)$ defined as the hyperspace of all nonempty subsets of X with at most n points. The homogeneity degree $hd(X)$ of a continuum X is the number of orbits for the action of the group of homeomorphisms of X onto itself. In this paper we determine $hd(F_n(X))$ for every manifold without boundary X and $n \in \mathbb{N}$. We also compute $hd(F_n[0, 1])$ for all $n \in \mathbb{N}$.

INTRODUCTION

A *continuum* is a nonempty compact connected metric space.

Here, the word *manifold* refers to a compact connected manifold with or without boundary.

Given a continuum X , the n^{th} -symmetric product of X is the hyperspace

$$F_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}.$$

The hyperspace $F_n(X)$ is considered with the Vietoris topology.

Given a continuum X , let $\mathcal{H}(X)$ denote the group of homeomorphisms of X onto itself. An *orbit* in X is a class of the equivalence relation in X given by p is equivalent to q if there exists $h \in \mathcal{H}(X)$ such that $h(p) = q$.

The *homogeneity degree*, $hd(X)$ of the continuum X is defined as

$$hd(X) = \text{number of orbits in } X.$$

When $hd(X) = n$, the continuum X is known to be $\frac{1}{n}$ -homogeneous, and when $hd(X) = 1$, X is *homogeneous*.

In [10], P. Pellicer-Covarrubias studied continua X for which $hd(F_2(X)) = 2$. Recently, I. Calderón, R. Hernández-Gutiérrez and A. Illanes [2] proved that if P is the pseudo-arc, then $hd(F_2(P)) = 3$. Other papers on homogeneity degrees of hyperspaces are [4] and [9].

In this paper we determine $hd(F_n(X))$ for every manifold without boundary X and $n \in \mathbb{N}$. We also compute $hd(F_n[0, 1])$ for all $n \in \mathbb{N}$. Since $F_1(X)$ is homeomorphic to X , $hd(F_1(X)) = hd(X)$. Thus, $hd(F_1(X)) = 1$ for every manifold without boundary X and $hd(F_1([0, 1])) = 2$.

MANIFOLDS WITHOUT BOUNDARY

We denote by S^1 the unit circle in the plane.

Given a continuum X , $n \in \mathbb{N}$ and subsets U_1, \dots, U_m of X , let

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$$\langle U_1, \dots, U_m \rangle_n = \{A \in F_n(X) : A \subset U_1 \cup \dots \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \dots, m\}\}.$$

Then the family $\{\langle U_1, \dots, U_m \rangle_n \subset F_n(X) : m \leq n \text{ and } U_1, \dots, U_m \text{ are open in } X\}$ is a basis for the Vietoris topology in $F_n(X)$ [7]. If A is any set and $n \in \mathbb{N}$, let $[A]^n = \{B \subset A : |B| = n\}$. Let Y be a topological space. A subset $Z \subset Y$ is a *topological type*, or just a *type*, if for each $h \in \mathcal{H}(Y)$, $h(Z) = Z$.

Lemma 1. *Let X be a locally connected continuum such that $[X]^4$ is a type in $F_4(X)$. Then for each $h \in \mathcal{H}(F_4(X))$, $h([X]^2) \cap [X]^3 = \emptyset$.*

Proof. Suppose to the contrary that there exist $h \in \mathcal{H}(F_4(X))$ and $A = \{a_1, a_2\} \in [X]^2$ such that $h(A) = B = \{b_1, b_2, b_3\} \in [X]^3$.

Let V_1, V_2 and V_3 be pairwise disjoint open connected sets of X such that $b_i \in V_i$ for each $i \in \{1, 2, 3\}$. Let U_1, U_2 be disjoint open connected subsets of X such that $a_1 \in U_1, a_2 \in U_2$ and $h(\langle U_1, U_2 \rangle) \subset \langle V_1, V_2, V_3 \rangle$.

The neighborhoods $\langle U_1, U_2 \rangle$ and $\langle V_1, V_2, V_3 \rangle$ have different topological structures. We will use this to arrive to a contradiction.

Let $\mathcal{U} = \langle U_1, U_2 \rangle$ and $\mathcal{V} = h(\mathcal{U})$.

First, we will describe the components of $\mathcal{U} \cap [X]^4$. For each $i \in \{1, 2, 3\}$, let

$$\mathcal{U}_i = \{A \in F_4(X) : |A \cap U_1| = i \text{ and } |A \cap U_2| = 4 - i\}.$$

Then it can be proved that the following properties hold.

- (U1) \mathcal{U}_i is a nonempty subset of $\mathcal{U} \cap [X]^4$ for each $i \in \{1, 2, 3\}$,
- (U2) $\mathcal{U} \cap [X]^4 = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$,
- (U3) $\text{cl}_{F_4(X)}(\mathcal{U}_i) \cap \mathcal{U}_j = \emptyset$ if $i, j \in \{1, 2, 3\}$ and $i \neq j$, and
- (U4) \mathcal{U}_i is arcwise connected for all $i \in \{1, 2, 3\}$.

Thus, it follows that $\mathcal{U}_1, \mathcal{U}_2$ and \mathcal{U}_3 are exactly the components of $\mathcal{U} \cap [X]^4$.

Now, for each $i \in \{1, 2, 3\}$, let

$$\mathcal{V}_i = \{A \in \mathcal{V} : |A \cap V_i| = 2\}.$$

The following properties hold.

- (V1) \mathcal{V}_i is a nonempty subset of $\mathcal{V} \cap [X]^4$ for each $i \in \{1, 2, 3\}$,
- (V2) $\mathcal{V} \cap [X]^4 = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, and
- (V3) $\text{cl}_{F_4(X)}(\mathcal{V}_i) \cap \mathcal{V}_j = \emptyset$ if $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Only (V1) requires further explanation. By hypothesis, $h(\mathcal{U} \setminus [X]^4) = \mathcal{V} \setminus [X]^4$ is non-empty. Choose any $E \in \mathcal{V} \setminus [X]^4$, clearly $E \in [X]^3$. Now let $i \in \{1, 2, 3\}$. Since \mathcal{V} is a neighborhood of E in $F_4(X)$, there exists $p_i \in U \setminus E$ such that $E \cup \{p_i\} \in \mathcal{V}$. Then $E \cup \{p_i\} \in \mathcal{V}_i$.

From property (V2), we have $h(\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3) = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$. Given $i \in \{1, 2, 3\}$, since $h(\mathcal{U}_i)$ is connected, there exists $k(i) \in \{1, 2, 3\}$ such that $h(\mathcal{U}_i) \subset \mathcal{V}_{k(i)}$. Since $h(\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3) = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, the function $k : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$, is surjective. Since $\mathcal{V}_1, \mathcal{V}_2$ and \mathcal{V}_3 play symmetric roles, we may assume that $h(\mathcal{U}_i) = \mathcal{V}_i$ for all $i \in \{1, 2, 3\}$. Since $\mathcal{V}_1, \mathcal{V}_2$ and \mathcal{V}_3 are pairwise disjoint, we obtain that indeed, $h(\mathcal{U}_i) = \mathcal{V}_i$ for all $i \in \{1, 2, 3\}$. So in fact, $\mathcal{V}_1, \mathcal{V}_2$ and \mathcal{V}_3 are the components of $\mathcal{V} \cap [X]^4$.

Let $C \in \mathcal{U}$ be such that $|C \cap U_1| = 2$ and $|C \cap U_2| = 1$. Denote $D = h(C)$. Since $C \in \mathcal{U} - [X]^4$ and $[X]^4$ is invariant under h , we obtain that $D \in \mathcal{V} - [X]^4$. Then $D \in \langle V_1, V_2, V_3 \rangle$, so $|D| = 3$. So note that we have the following properties.

- (a) There is a neighborhood \mathcal{R} of C such that $\mathcal{R} \cap \mathcal{U}_1 = \emptyset$ and

(b) If \mathcal{S} is a neighborhood of D , then $\mathcal{S} \cap \mathcal{V}_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$.

Clearly, property (a) contradicts property (b) since $h(\mathcal{U}_1) = \mathcal{V}_1$. This contradiction shows that such a homeomorphism h does not exist. \square

Lemma 2. *Let X be an m -manifold (with or without boundary) and $1 \leq k \leq n$. Suppose that either: (a) $m \geq 2$ and $n \geq 3$; or (b) $m = 1$ and $n \geq 4$. Then the set $[X]^k$ is a type in $F_n(X)$.*

Proof. Let $h \in \mathcal{H}(F_n(X))$.

Define $\mathcal{D}_n(X) = \{A \in F_n(X) : A \text{ has a neighborhood in } F_n(X) \text{ that is an } nm\text{-cell}\}$.

Since the definition of $\mathcal{D}_n(X)$ is given in terms of topological concepts, we have $h(\mathcal{D}_n(X)) = \mathcal{D}_n(X)$.

In the case that $m \geq 2$ and $n \geq 3$, Theorem 17 in [3] implies that $h(F_1(X)) = F_1(X) = [X]^1$. Moreover, by the first part of the proof of Theorem 17 in [3], we have $\mathcal{D}_n(X) = [X]^n$. Thus, $h([X]^n) = [X]^n$. This implies that $h(F_{n-1}(X)) = F_{n-1}(X)$. If $3 \leq n-1$ we can repeat the argument to obtain that $h([X]^{n-1}) = [X]^{n-1}$. Proceeding in this way, we have that for each $k \geq 3$, $h([X]^k) = [X]^k$. Hence, $h(F_2(X)) = F_2(X)$. Since $h(F_1(X)) = F_1(X)$, we are done.

In the case that $m = 1$ and $n \geq 4$, we have that X is either an arc or a simple closed curve.

By Corollary 4.4 in [6], $\mathcal{D}_n(X) = [X]^n$. Thus, $h([X]^n) = [X]^n$. Proceeding as before, we obtain that for each $k \geq 4$, $h([X]^k) = [X]^k$. Thus, $h(F_3(X)) = F_3(X)$. Moreover, by Corollary 7 in [3], we have $h(F_1(X)) = F_1(X)$. Finally, Lemma 1 implies that $h([X]^2) = [X]^2$ and $h([X]^3) = [X]^3$. \square

The following lemma is well known.

Lemma 3. *Let X be a manifold without boundary and $n \in \mathbb{N}$. Then for every $A, B \in [X]^n$, there exists $h \in \mathcal{H}(X)$ such that $h(A) = B$.*

Corollary 4. *Let X be a manifold without boundary and $n \geq 2$. Then $hd(F_n(X)) \leq n$.*

Applying Lemmas 2 and 3, we conclude the following theorem.

Proposition 5. *Let X be an m -manifold without boundary with $m \geq 2$ and let $n \geq 3$. Then $hd(F_n(X)) = n$.*

Given a continuum X and $n \geq 2$, the hyperspace $F_n(X)$ is rigid provided that for each $h \in \mathcal{H}(F_n(X))$, $h(F_1(X)) = F_1(X)$. Rigidity of symmetric products was studied in [3], where it was shown that ([3, Theorem 17]) if X is an m -manifold, $m \geq 2$ and $n \geq 3$, then $F_n(X)$ is rigid. Using a theorem by R. Molski [8], we show that this result is also true for $m \geq 3$ and $n = 2$.

Lemma 6. *Let X be an m -manifold with $m \geq 3$. Then $F_2(X)$ is rigid.*

Proof. Let $\mathcal{D}_2(X) = \{A \in F_2(X) : A \text{ has a neighborhood in } F_2(X) \text{ embeddable in } \mathbb{R}^{2m}\}$. Clearly, each element $A \in [X]^2$ has a neighborhood homeomorphic to $[0, 1]^{2m}$, so $[X]^2 \subset \mathcal{D}_2(X)$.

Given $p \in X$, let \mathcal{M} be a neighborhood of $\{p\}$ in $F_2(X)$. Then there exists an open subset U of X such that $p \in U$ and $\langle U \rangle_2 \subset \mathcal{M}$. Let R be an m -cell such that $R \subset U$. Then $F_2(R) \subset \langle U \rangle_2 \subset \mathcal{M}$. By Theorem 3 in [8], $F_2(R)$ (and then \mathcal{M}) cannot be embedded in \mathbb{R}^{2m} . We have shown that $\mathcal{D}_2(X) \subset [X]^2$. Therefore,

$\mathcal{D}_2(X) = [X]^2$. Since the definition of $\mathcal{D}_2(X)$ is given in topological terms, we conclude that $F_2(X)$ is rigid. \square

Combining Corollary 4 and Lemma 6, we obtain the following result.

Proposition 7. *Let X be an m -manifold without boundary with $m \geq 3$. Then $hd(F_2(X)) = 2$.*

By the Corollary to Theorem 1 in [8], if X is a 2-manifold without boundary, then $F_2(X)$ is a 4-manifold without boundary, hence $hd(F_2(X)) = 1$.

Proposition 8. *Let $n \in \mathbb{N}$. Then $hd(F_3(S^1)) = 1$, and if $n \neq 3$, then $hd(F_n(S^1)) = n$.*

Proof. By [1], $F_3(S^1)$ is homeomorphic to the unit sphere in the Euclidean space \mathbb{R}^4 . So, $hd(F_3(S^1)) = 1$. It is well known that $F_2(S^1)$ is homeomorphic to the Moebius strip (see section 14 in [5]). Thus, $hd(F_2(S^1)) = 2$. Clearly, $hd(F_1(S^1)) = 1$. Finally, if $n \geq 4$, by Lemmas 2 and 3, $hd(F_n(S^1)) = n$. \square

We summarize the results of this section in the following theorem.

Theorem 9. *Let X be an m -manifold without boundary and $n \in \mathbb{N}$. Then*

- (a) *if either $m \geq 2$ and $n \neq 2$ or $m = 1$ and $n \neq 3$, then $hd(F_n(X)) = n$,*
- (b) *if $m = 2$ (X is a surface), then $hd(F_2(X)) = 1$, and*
- (c) *if $m = 1$ (X is a simple closed curve) and $n = 3$, then $hd(F_2(X)) = 1$.*

In the case that X is a manifold with boundary, it seems to be difficult to give a result so precise as Theorem 9. The following example shows that $hd(F_n(X))$ could depend not only on n but in the number of components of the manifold boundary of X .

For each $k \in \mathbb{N}$, we can consider a family of disjoint subsets T_1, T_2, \dots, T_k of \mathbb{R}^3 , where each T_i is a 2-sphere with i handles and if $i \neq j$, then T_i is contained in the unbounded domain of $\mathbb{R}^3 - T_j$. Suppose that M is a closed 3-ball in \mathbb{R}^3 containing $T_1 \cup \dots \cup T_k$ in its interior. Consider the continuum X which is the closure of the intermediate region bounded by M and $T_1 \cup \dots \cup T_k$. Clearly, X is a manifold with boundary.

By Theorem 17 of [3], if $n \geq 3$, then $F_n(X)$ is rigid. So, if $h \in \mathcal{H}(F_n(X))$, then $h(F_1(X)) = F_1(X)$. Thus, $h|_{F_1(X)} : F_1(X) \rightarrow F_1(X)$ is a homeomorphism. This implies that $h(T_i) = T_i$. Therefore, for each $n \geq 3$, $hd(F_n(X))$ is greater than k .

THE UNIT INTERVAL

Lemma 10. *Let $n \geq 2$ and $A, B \in F_n([0, 1])$ be such that $|A| = |B| \geq 2$ and $A \cap \{0, 1\} \neq \emptyset \neq B \cap \{0, 1\}$. Then there is a homeomorphism $h_0 \in \mathcal{H}(F_n([0, 1]))$ such that $h_0(B) = A$.*

Proof. Suppose that $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_m\}$, where $|A| = |B| = k \geq 2$, $a_1 < \dots < a_m$ and $b_1 < \dots < b_m$. In the cases:

- (a) $a_1 = 0 = b_1$ and $a_m = 1 = b_m$,
- (b) $a_1 = 0 = b_1$ and $\max\{a_m, b_m\} < 1$,
- (c) $0 < \min\{a_1, b_1\}$ and $a_m = 1 = b_m$,
- (d) $0 = a_1$, $a_m < 1$, $0 < b_1$ and $b_m = 1$, and
- (e) $0 < a_1$, $a_m = 1$, $0 = b_1$ and $b_m < 1$,

it is easy to show that there is $g \in \mathcal{H}([0, 1])$ such that $g(B) = A$. Then the induced mapping $h_0 = F_n(g) : F_n([0, 1]) \rightarrow F_n([0, 1])$ satisfies that $h_0(B) = A$. The rest of the cases are similar to the following one.

(f) $a_1 = 0$, $a_m < 1$, $b_1 = 0$ and $b_m = 1$.

Thus, we only need to show case (f).

In case (f), for each nonempty closed subset D of $[0, 1]$, let $m(D) = \min D$ and $M(D) = \max D$. Consider the mapping $\varphi : F_n([0, 1]) \rightarrow \mathbb{R}^2$ given by $\varphi(D) = (m(D), M(D))$. Clearly, φ is a mapping whose image is the triangle T in the plane \mathbb{R}^2 with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. Let Δ be the convex segment in T with end points $(0, 0)$ and $(1, 1)$. Clearly, there exists $\sigma = (\sigma_1, \sigma_2) \in \mathcal{H}(T)$ such that $\sigma|_\Delta$ is the identity in Δ and $\sigma(0, 1) = (0, a_m)$. Notice that if $(x, y) \in T$ and $x < y$, then $\sigma_1(x) < \sigma_2(y)$.

Given two nondegenerate subintervals J and K of $[0, 1]$, let $\psi(J, K) : J \rightarrow K$ be the strictly increasing linear homeomorphism, that is, for each $t \in J$,

$$\psi(J, K)(t) = \frac{t - m(J)}{M(J) - m(J)}M(K) + \frac{M(J) - t}{M(J) - m(J)}m(K).$$

Define $h : F_n([0, 1]) \rightarrow F_n([0, 1])$ by

$$h(D) = \begin{cases} \psi([m(D), M(D)], [\sigma_1(\varphi(D)), \sigma_2(\varphi(D))])(D), & \text{if } D \notin F_1([0, 1]), \\ D, & \text{if } D \in F_1([0, 1]). \end{cases}$$

Clearly, h is continuous in the open set $F_n([0, 1]) - F_1([0, 1])$. To complete the proof that h is continuous, let $\{D_k\}_{k=1}^\infty$ be a sequence in $F_n([0, 1]) - F_1([0, 1])$ converging to an element $\{p\} \in F_1([0, 1])$. Then $\lim_{k \rightarrow \infty} M(D_k) = p = \lim_{k \rightarrow \infty} m(D_k)$, $\lim_{k \rightarrow \infty} \varphi(D_k) = (p, p)$ and $\lim_{k \rightarrow \infty} \sigma(\varphi(D_k)) = (p, p)$. Since for each $k \in \mathbb{N}$, $\psi([m(D_k), M(D_k)], [\sigma_1(\varphi(D_k)), \sigma_2(\varphi(D_k))])(D_k)$ is a subset of $[\sigma_1(\varphi(D_k)), \sigma_2(\varphi(D_k))]$ and $\lim_{k \rightarrow \infty} [\sigma_1(\varphi(D_k)), \sigma_2(\varphi(D_k))] = \{p\}$, we obtain that

$$\lim_{k \rightarrow \infty} \psi([m(D_k), M(D_k)], [\sigma_1(\varphi(D_k)), \sigma_2(\varphi(D_k))])(D_k) = \{p\}.$$

Thus, $\lim_{k \rightarrow \infty} h(D_k) = h(\{p\})$. Therefore, h is continuous.

In order to show that h is one-to-one, let $E, D \in F_n([0, 1])$ be such that $h(D) = h(E)$.

In the case $D \notin F_1([0, 1])$, $m(D) < M(D)$, so $\varphi(D) \notin \Delta$ and $\sigma(\varphi(D)) \notin \Delta$. So, $\sigma_1(\varphi(D)) < \sigma_2(\varphi(D))$. Since $\psi([m(D), M(D)], [\sigma_1(\varphi(D)), \sigma_2(\varphi(D))])(m(D)) = \sigma_1(\varphi(D))$ and $\psi([m(D), M(D)], [\sigma_1(\varphi(D)), \sigma_2(\varphi(D))])(M(D)) = \sigma_2(\varphi(D))$, we obtain that $m(h(D)) = \sigma_1(\varphi(D))$, $M(h(D)) = \sigma_2(\varphi(D))$. So, $h(D)$ is nondegenerate. Thus, $h(E)$ is nondegenerate, $m(h(E)) = \sigma_1(\varphi(D))$, $M(h(E)) = \sigma_2(\varphi(D))$ and $E \notin F_1([0, 1])$. Similarly, $m(h(E)) = \sigma_1(\varphi(E))$ and $M(h(E)) = \sigma_2(\varphi(E))$. Hence, $\sigma(\varphi(E)) = \sigma(\varphi(D))$, $\varphi(E) = \varphi(D)$, $m(E) = m(D)$ and $M(E) = M(D)$. Thus,

$$\begin{aligned} \psi([m(D), M(D)], [\sigma_1(\varphi(D)), \sigma_2(\varphi(D))]) &= \\ \psi([m(E), M(E)], [\sigma_1(\varphi(E)), \sigma_2(\varphi(E))]) & \end{aligned}$$

Since this mapping is one-to-one, we conclude that $E = D$.

In the case that $D \in F_1([0, 1])$, proceeding as before, we obtain that $E \in F_1([0, 1])$. Thus, $D = h(D) = h(E) = E$. Therefore, h is one-to-one.

Now we check that h is onto. Take $E \in F_n([0, 1])$. If $E \in F_1([0, 1])$, then $E = h(E)$ and we are done. So, we suppose that $E \notin F_1([0, 1])$. Let $x = m(E)$, $y = M(E)$ and $(a, b) = \sigma^{-1}(x, y)$. Then $x < y$ and $a < b$. Since $\psi([a, b], [x, y]) : [a, b] \rightarrow$

$[x, y]$ is onto and $E \subset [x, y]$, we can define $D = (\psi([a, b], [x, y]))^{-1}(E) \in F_n([0, 1])$. Note that $D \notin F_1([0, 1])$ and $h(D) = E$. Therefore, h is onto.

We have shown that h is a homeomorphism.

Note that $h(B) = \psi([0, 1], [\sigma_1(0, 1), \sigma_2(0, 1)])(B) \subset [\sigma_1(0, 1), \sigma_2(0, 1)] = [0, a_m]$ and $\{0, a_m\} \subset h(B)$. Thus, A and $h(B)$ are as in (b). So, there exists a homeomorphism $h_1 \in \mathcal{H}(F_n([0, 1]))$ such that $h_1(h(B)) = A$. Therefore, $h_0 = h_1 \circ h$ is a homeomorphism such that $h_0(B) = A$. \square

Theorem 11. *Let $n \in \mathbb{N}$. Then:*

- (a) *if $n \notin \{2, 3\}$, then $hd(F_n([0, 1])) = 2n$, and*
- (b) *if $n \in \{2, 3\}$, then $hd(F_n([0, 1])) = 2$.*

Proof. (a) Since $F_1([0, 1])$ is homeomorphic to $[0, 1]$, $hd(F_1([0, 1])) = 2$. Let $n \geq 4$, $1 \leq k \leq n$ and $h \in \mathcal{H}(F_n([0, 1]))$. By Lemma 2, $h([0, 1]^k) = [0, 1]^k$. Note that $[0, 1]^k = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1 = \{A \in [0, 1]^k : A \text{ has a neighborhood } \mathcal{M} \text{ in } [0, 1]^k \text{ that is a } k\text{-cell and } A \text{ is in the manifold boundary of } \mathcal{M}\}$ and $\mathcal{D}_2 = \{A \in [0, 1]^k : A \text{ has a neighborhood } \mathcal{M} \text{ in } [0, 1]^k \text{ that is a } k\text{-cell and } A \text{ is not in the manifold boundary of } \mathcal{M}\}$. Clearly, $h(\mathcal{D}_1) = \mathcal{D}_1$ and $h(\mathcal{D}_2) = \mathcal{D}_2$. By Lemma 10, \mathcal{D}_2 is an orbit in $F_n([0, 1])$. Clearly, \mathcal{D}_1 is an orbit in $F_n([0, 1])$. Therefore, $F_n([0, 1])$ contains exactly $2n$ orbits.

By sections 13 and 14 in [5], $F_2([0, 1])$ is a 2-cell and $F_3([0, 1])$ is a 3-cell. This implies (b). \square

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